Perturbation theory

Start: two diffusion equations characterized by Σ_a, ϕ, k and Σ'_a, ϕ', k' respectively.

Start with two steady-state diffusion equations.

$$-D\nabla^2\phi + \Sigma_a\phi = \frac{1}{k}\nu\Sigma_f\phi$$

$$-D\nabla^2\phi' + \Sigma'_a\phi' = \frac{1}{k'}\nu\Sigma_f\phi'$$

Integrate each equation over the complementary flux (assume 1D).

$$\int -D\nabla^2\phi\phi' dx + \int \Sigma_a\phi\phi' dx = \int \frac{1}{k}\nu\Sigma_f\phi\phi' dx$$

$$\int -D\nabla^2\phi'\phi dx + \int \Sigma'_a\phi'\phi dx = \int \frac{1}{k'}\nu\Sigma_f\phi'\phi dx$$

Subtract the second equation from the first.

$$\int \Sigma_a\phi\phi' dx - \int \Sigma'_a\phi'\phi dx = \int \frac{1}{k}\nu\Sigma_a\phi\phi' dx - \int \frac{1}{k'}\nu\Sigma_f\phi'\phi dx$$

Define $\delta\Sigma_a = \Sigma_a - \Sigma'_a$ and combine like terms.

$$\int \delta\Sigma_a\phi\phi' dx = \left(\frac{1}{k} - \frac{1}{k'}\right)\int \nu\Sigma_f\phi\phi' dx$$

Consider $\Delta\rho = \rho' - \rho$

$$\Delta\rho = \rho' - \rho = \frac{k'-1}{k'} - \frac{k-1}{k} = \frac{(k'-1)k - (k-1)k'}{kk'} = \frac{k'k - k - k'k + k'}{kk'} = \frac{1}{k} - \frac{1}{k'}$$

Solve for $\Delta\rho$

$$\Delta\rho = \frac{\int \delta\Sigma_a\phi\phi' dx}{\int \nu\Sigma_f\phi\phi' dx}$$